International Journal of Theoretical Physics, Vol. 33, No. 4, 1994

3-Manifolds for Relativists

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Received September 20, 1993

In canonical quantum gravity certain topological properties of 3-manifolds are of interest. This article gives an account of those properties which have so far received sufficient attention, especially those concerning the diffeomorphism groups of 3-manifolds. We give a summary of these properties and list some old and new results concerning them. The appendix contains a discussion of the group of large diffeomorphisms of the l-handle 3-manifold.

INTRODUCTION

In the canonical formulation of general relativity and in particular in all approaches to canonical quantum gravity it emerges that the diffeomorphism groups of closed 3-manifolds are of particular interest. Here one is interested in a variety of questions concerning either the whole diffeomorphism group or certain subgroups thereof. More precisely, one is, e.g., interested in whether the 3-manifold admits orientation-reversing self-diffeomorphisms (Sorkin, 1989) (following this reference we will call a manifold chiral iff it admits no such diffeomorphism) or what diffeomorphisms not connected to the identity there are in the subgroup fixing a frame (Friedman and Sorkin, 1980; Isham, 1981; Sorkin, 1989; Witt, 1986). Other topological invariants of this latter subgroup are also argued to be of interest in quantum gravity (Giulini, 1992*a*). In the sum-over-histories approach one is also interested in whether a given 3-manifold can be considered as the spatial boundary of a spin-Lorentz 4-manifold (Gibbons and Hawking, 1992; Giulini, 1993) [following Giulini (1993) such 3-manifolds will be called nuclear]. Motivations for studying these questions may be found in the cited literature and references therein.

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The purpose of this paper is to present an account of results (partially new) on these questions in a comprehensive way. A simple but nontrivial example on which a typical diffeomorphisms group can be studied is presented in the appendix. *Throughout the paper we shall take the term 3-manifold to also imply closed, connected, and oriented unless stated otherwise.* Since 3-manifolds are not classified, we follow the standard route by presenting the results for some prime (defined below) 3-manifolds and indicate how the quantity in question behaves under taking connected sums (defined below). Let us now recall some basic facts from the subject of 3-manifolds. A standard textbook is Hempel (1976), which also contains all the relevant references which we do not list separately.

Definition 1. Let Σ, Σ_1, and Σ_2 be 3-manifolds. We say that Σ is the connected sum of Σ_1 and Σ_2, in symbols: $\Sigma = \Sigma_1 \uplus \Sigma_2$, if the following conditions are satisfied: There exist two open 3-balls, $B_i \subset \Sigma_i$, and orientation-preserving embeddings, $h_i: R_i \to \Sigma$, where $R_i = \Sigma_i - B_i$, such that $h_1(R_1) \cap h_2(R_2) = h_1(\partial R_1) = h_2(\partial R_2)$ and $h_1(R_1) \cup h_2(R_2) = \Sigma$.

The operation of taking the connected sum is well defined, commutative, and associative. Taking the connected sum of any 3-manifold Σ with the 3-sphere results in a 3-manifold diffeomorphic to Σ.

Definition 2. A 3-manifold is called a prime 3-manifold (or simply prime) iff it is not the connected sum of two 3-manifolds none of which is diffeomorphic to the 3-sphere.

Theorem 1 (Kneser). Every 3-manifold $\Sigma \neq S^3$ is diffeomorphic to a finite connected sum of prime 3-manifolds $\Pi_1, \ldots, \Pi_n$ with $\Pi_i \neq S^3$ for all i:

$$\Sigma = \biguplus_{i=1}^{n} \Pi_i$$

Theorem 2 (Milnor). This decomposition is unique, in the sense that for any other decomposition of Σ into prime 3-manifolds $\{\Pi'_1, \ldots, \Pi'_m\}$ with $\Pi'_i \neq S^3$ for all i, it follows that $n = m$ and that there exist orientation-preserving diffeomorphisms $\phi_i: \Pi_i \to \Pi'_{\sigma(i)}$, $i = 1, \ldots, n$, for some permutation σ.

Note that since the manifolds are all oriented, a prime-manifold Π and its oppositely oriented version $(-\Pi)$ form different primes (in the sense of Theorem 2) iff Π_i is chiral.

A 3-manifold Σ is called *irreducible* if every embedded 2-sphere in Σ bounds a 3-ball. Clearly, an irreducible 3-manifold is prime. The converse is almost true with only one exception: The "handle" 3-manifold, $S^1 \times S^2$, is the only nonirreducible prime. Irreducibility also implies that the 3-manifold has trivial second homotopy group. [Proof: The so called sphere-theorem (see, e.g., Hempel, 1976) implies for a nontrivial second homotopy group the existence of an element different from the identity which can be represented by an embedded sphere. However, irreducibility enforces each embedded sphere to be contractible.]

1. PROPERTIES OF THE DIFFEOMORPHISM GROUP

Associated to a manifold Σ we introduce the group of C^∞-diffeomorphisms and several subgroups thereof. To define these, let $\infty \in \Sigma$ denote a fixed, preselected point.[2] We define

$$D(\Sigma) := \{\phi = C^\infty\text{-diffeomorphism of } \Sigma\} \tag{1.1a}$$

$$D_\infty(\Sigma) := \{\phi \in D(\Sigma)/\phi(\infty) = \infty\} \tag{1.1b}$$

$$D_F(\Sigma) := \{\phi \in D_\infty(\Sigma)/\phi_*|_\infty = id\} \tag{1.1c}$$

where $\phi_*|_\infty\colon T_\infty(\Sigma) \to T_\infty(\Sigma)$ is the differential at ∞ of the map ϕ. In words, D contains all diffeomorphisms, D_∞ those fixing ∞, and D_F those which in addition fix the frames at ∞. There are clearly no orientation-reversing diffeomorphisms in D_F, but we take D and D_∞ to include orientation-reversing diffeomorphisms, should they exist. Additional superscripts $+$ or 0 may then refer to the normal subgroup of orientation-preserving diffeomorphisms or the identity component, respectively. Note also that D_F is a normal subgroup of D_∞, but D_∞ is no normal subgroup of D.

More subgroups of D may be introduced in the following way: Let $N \subset \Sigma$ be a closed subset. Then

$$D(\Sigma, \text{rel } N) = \{\phi \in D(\Sigma)/\phi|_N = id\} \tag{1.1d}$$

denotes the diffeomorphisms fixing N [thus generalizing (1.1b)]. If B_ϵ denotes any closed 3-disc neighborhood of ∞ with diameter ϵ (in some fiducial metric), then clearly $D(\Sigma, \text{rel } B_\epsilon) \subset D_F(\Sigma)\ \forall\epsilon$ and one may, loosely speaking, regard $D_F(\Sigma)$ as a limit of $D(\Sigma, \text{rel } B_\epsilon)$ for $\epsilon \to 0$. If one is only interested in the topological features of the diffeomorphism groups, as we are, one may indeed replace any $D(\Sigma, \text{rel } B_\epsilon)$ with $D_F(\Sigma)$, or vice versa. Now, it is in fact $D_F(\Sigma)$ [or, equivalently, $D(\Sigma, \text{rel } B_\epsilon)$] which is of primary interest in general relativity [we refer to Giulini (1992*a*) for a deeper discussion of this point] and to the study of which we now turn.

The fundamental theorem, first announced in Rourke and de Sá (1974), then elaborated on in Hendriks and Laudenbach (1984), Hendriks and

[2]The reason for this name is that in the study of asymptotically isolated (flat) systems, where the underlying 3-manifold is noncompact with one asymptotic region, it is convenient to work with the one-point compactification by adding a point at infinity.

McCullough (1987), and McCullough (1986), aims to make precise how the diffeomorphisms $D_F(\Sigma)$ for any nonprime 3-manifold Σ are built up from those of its prime constituents plus extra ones, and how these extra ones can be generated by certain basic operations which allow for more or less intuitive geometric interpretations. In order to fully appreciate this result, we need to go through some details and preliminary results. At the end, when grouped together, these form what we call the theorem of Rourke and de Sá, although the proofs of the claims in Rourke and de Sá (1974) given in Hendriks and Laudenbach (1984) and Hendriks and McCullough (1987) do not seem to have been given in a complete form by Rourke and de Sá (1974). In what follows, D's and S's with lower-case indices i, j, k denote open 3-disks and 2-spheres, respectively. the irreducible primes are notationally separated from the reducible one, $S^1 \times S^2$, and will be denoted by P_i.

Let

$$\Sigma = \left(\biguplus_{i=1}^{n} P_i \right) \uplus \left(\biguplus_{j=1}^{l} S^2 \times S^1 \right) \tag{1.2}$$

be constructed in the following way: First, take $P_i' = P_i - D_i$, $1 \leq i \leq n$, $\partial P_i' = S_i'$, and l copies of $I \times S^2$, labeled $I \times S_j^2$, $i \leq j \leq l$, such that $\partial(I \times S_j^2) = S_{j,1}' \cup S_{j,2}'$. We call them the factors of Σ. Second, take a closed 3-disc D ($\partial D = S_0$) and remove $n + 2l$ mutually disjoint open 3-discs:

$$\begin{aligned} B &:= D - \left\{ \bigcup_{i=1}^{n} D_r \cup \bigcup_{(j,k)=(1,1)}^{(l,2)} D_{(j,k)} \right\} \\ \partial B &= S_0 \cup \bigcup_{i=1}^{n} S_i \cup \bigcup_{(j,k)=(1,1)}^{(l,2)} S_{(j,k)} \end{aligned} \tag{1.3}$$

We shall call B the base and S_0 the sphere at ∞.[3] Third, eliminate all but S_0 of the boundary components of ∂B by gluing the boundaries of the factors onto the boundary components of the base by using some identification diffeomorphisms

$$G_i\colon\ S_i' \to S_i \qquad \text{and} \qquad G_{(i,k)}\colon\ S_{(i,k)}' \to S_{(i,k)} \tag{1.4}$$

such that the resulting space carries an orientation which is compatible with the individual orientations given to the factors and the base beforehand. For this, the maps in (1.4) must be orientation-reversing with respect to the induced orientations. This construction is unambiguous since any two diffeomorphisms of 2-spheres which either preserve or reverse orientation are isotopic. Finally, we cap-off S_0 with a 3-disk D_0 to obtain Σ. The

[3]Since, in the notation above, it will be taken as the boundary ∂B_ϵ of a neighborhood of ∞.

point ∞ may now be taken as the center of the disc D_0 and D_0 itself may be identified with one of the B_c's mentioned above.

Using this decomposition, we can now define four classes of diffeomorphisms of Σ that will suffice to generate all orientation-preserving diffeomorphism up to isotopy (Theorem 3 below). A proof may be found in McCullough (1986). To unify the notation, we shall commit a slight abuse of language in not always distinguishing between $D_F(\Sigma)$ and $D(\Sigma, \text{rel } D_0)$ or $D_F(P_i)$ and $D(P'_i, \text{rel } S'_i)$.

1. *Internal diffeomorphisms*: These are diffeomorphisms that reduce to the identity when restricted to B. The individual supports P'_i and $I \times S^2_i$ are disjoint in Σ and elements with support in two disjoint sets of factors clearly commute. For the handles $S^1 \times S^2$ one can show that $D(I \times S^2_i, \text{rel } \partial I \times S^2_i)$ is homotopy equivalent to the fiber-preserving diffeomorphisms not exchanging the boundary components ($I \times S^2$ being viewed as the trivial S^2 bundle over I), and hence to $\Omega(SO(3))$, the space of based loops of $SO(3)$. We write

$$D_{\text{int}} = \prod_{i=1}^{n} D_F(P_i) \times \prod_{j=1}^{l} \Omega(SO(3)) \tag{1.5}$$

2. *Exchange diffeomorphisms*: Given any two diffeomorphic factors P'_i, P'_j (resp. any two $I \times S^2_i, I \times S^2_j$) and the associated diffeomorphism $\phi_{ji}: P'_i \to P'_j$ (resp. $I \times S^2_i \to I \times S^2_j$). Given also a diffeomorphism ψ_{ji} of B, exchanging S_i and S_j (resp. $S_{(i,k)}$ and $S_{(j,k)}$ for $k = 1, 2$) in such a way that outside some neighborhood in B containing these but no other boundary spheres ψ_{ij} restricts to the identity. Let it also be adjusted in such a way that it be compatible with the gluing instructions (1.4):

$$G_j \circ \phi_{ji}|_{S'_i} = \psi_{ji}|_{S_i} \circ G_i \qquad \text{resp.} \qquad G_{(j,k)} \circ \phi_{ji}|_{S'_{(i,k)}} = \psi_{ji}|_{S_{(i,k)}} \circ G_{(i,k)} \tag{1.6}$$

Simultaneously performing ϕ_{ji} and ψ_{ji} now defines a diffeomorphism of Σ which we call an exchange of P_i and P_j (resp. $I \times S_i$ and $I \times S_j$). Any diffeomorphism generated by the exchanges of the type just described is then called an exchange diffeomorphism of Σ.

3. *Spin diffeomorphisms*: These are like the exchange diffeomorphisms, but concern only the two ends of handles. More precisely, take an orientation-preserving diffeomorphism $\phi_i: I \times S^2_i \to I \times S^2_i$, exchanging $S'_{(i,1)}$ and $S'_{(i,2)}$, and a diffeomorphism ψ_i of B exchanging $S_{(i,1)}$ and $S_{(i,2)}$ such that outside some neighborhood in B containing these but no other boundary spheres ψ_i restricts to the identity. Let it also be adjusted in such a way that it be compatible with the gluing instructions:

$$G_{(i,1)} \circ \phi_i|_{S'_{(i,2)}} = \psi_i|_{S_{(i,2)}} \circ G_{(i,2)} \qquad \text{and} \qquad G_{(i,2)} \circ \phi_i|_{S'_{(i,1)}} = \psi_i|_{S_{(i,1)}} \circ G_{(i,1)} \tag{1.7}$$

Performing ϕ_i and ψ_i simultaneously defines a diffeomorphism of Σ which we call a spin of the ith handle. All these generate the spin diffeomorphisms of Σ.

4. *Slide diffeomorphisms*: The diffeomorphisms mentioned so far leave B invariant as a set. Slides represent those elements that mix the interior of the factors with the base B. Consider a fixed factor P_i' (resp. $I \times S_i^2$) and a nonintersecting, noncontractible and oriented loop γ in the complement of all other prime factors through $p \in B$ (γ thus represents a nontrivial element in the obvious subgroup $\pi_1(P_i)$ [resp. $\pi_1(S^1 \times S^2)$] of $\pi_1(\Sigma)$). Now choose $j \neq i$, cut γ at p, and connect the two ends to two different (say antipodal) points of S_j so that the curve is still nonintersecting. A thin, closed neighborhood N of this 'curve-attached-to-sphere' has the topology of a solid 2-torus with an open ball removed from its interior. N has two boundary components, the two-sphere S_j as inner boundary, and a two-torus T as other boundary. An inner collar neighborhood T' of T, denoted by $[0, 1] \times T$ such that $1 \times T = T$, can be coordinatized by (t, θ, φ), where φ coordinate lines are running "parallel" to γ [i.e., generate $Z = \pi_1(N)$], and θ runs along the meridians (i.e., they are contractible within N). Let $\sigma: [0, 1] \to [0, 1]$ be a C^∞-function, such that $\sigma(0) = 0$, $\sigma(1) = 1$, and with vanishing derivatives at 0 and 1 to all orders. We define the following diffeomorphism on T':

$$s: \quad (t, \theta, \varphi) \mapsto (t', \theta', \varphi') := (t, \theta, \varphi + 2\pi\sigma(t)) \tag{1.8}$$

and continue it to the complement of T' by the identity. This defines a diffeomorphism of Σ with support in $T' \subset N$ which we call a slide of P_j' along γ. Analogously, instead of the sphere S_j, we could have taken any of the spheres $S_{(j,k)}$ [$j \neq i$ if γ generates $\pi_1(S^1 \times S_i)$]. In this case the resulting diffeomorphism is called a slide of the kth end of the jth handle, $I \times S_j^2$, along γ. We restricted attention to those γ that are homotopically nontrivial within Σ, since it may be shown that slides along contractible γ are isotopic to the identity and that slides along the composite loop $\gamma_1\gamma_2$ are isotopic to the composition of each individual slide. This ends our presentation of the four classes of diffeomorphisms.

Another important class of diffeomorphisms is given by the *rotations parallel to spheres*, which we now define. Given a 3-manifold Σ, an embedding $E: [0, 1] \times S^2 \to \Sigma$, a smooth, noncontractible loop $\lambda: [0, 1] \to SO(3)$ based at the identity, and a function σ as just defined for slides. On $[0, 1] \times S^2$ one has the diffeomorphism

$$r: \quad [0, 1] \times S^2 \to [0, 1] \times S^2, \qquad (t, x) \mapsto r(t, x) := (t, (\lambda \circ \sigma(t)) \cdot x) \tag{1.9}$$

through which we define a diffeomorphism of Σ (Im = Image)

$$R: \quad \Sigma \to \Sigma, \quad p \mapsto \begin{cases} E \circ r \circ E^{-1}(p) & \text{for } p \in \operatorname{Im} E \\ p & \text{for } p \notin \operatorname{Im} E \end{cases} \tag{1.10}$$

which we call *a rotation parallel to the spheres* $E(0)$ and $E(1)$. Any other choice of a smooth, noncontractible loop λ would give rise to an isotopic diffeomorphism. If Im E is a collar neighborhood of a sphere S in Σ, we may simply speak of a *rotation parallel to S*.

From (1.5) we see that a rotation parallel to a sphere $t \times S^2$ defines a diffeomorphism of the handle-manifold which is not connected to the identity. We call it a *belt-twist*. Applied to a particular factor $I \times S_i^2$ of Σ, it is isotopic to a rotation parallel to $S_{(i,1)}$ or $S_{(i,2)}$, which we consider as internal diffeomorphisms. In the same way, a rotation parallel to the connecting sphere S_i of P_i is considered as internal diffeomorphisms. We call it a *rotation of* P_i. We can also define a rotation of the ith handle by a rotation parallel to a sphere enclosing $S_{(i,1)}$ and $S_{(i,2)}$. However, this is easily seen to be isotopic to two rotations parallel to each one of these spheres (see, e.g., Giulini, 1992*a*) and hence isotopic to the identity within $D_F(\Sigma)$.

We say that Σ is *spinorial* iff a rotation parallel to S_0—which we simply call *a rotation of* Σ—is not connected to the identity in $D_F(\Sigma)$. A single handle manifold is thus not spinorial. It is easy to see that a rotation of Σ is isotopic to rotating all P_i. Thus, Σ is spinorial iff any of the P_i is (Giulini, 1992*a*).

Given these definitions, we remark that there is a certain ambiguity in the definition of slides of ends of handles or spinorial primes. To see this, note that an alternative choice of the map s in (1.8) would have been to let also θ wind once around a full range: $\theta' = \theta + 2\pi\sigma(t)$. This would impose an additional rotation parallel to the sphere boundary component S_j of N, so that the resulting diffeomorphism differs from the previous one by the rotation parallel to S_j, which may be thought of as an internal diffeomorphism and which is isotopic to the identity in $D(P_j', \mathrm{rel}\, S_j')$ iff P_j is a nonspinorial prime. In the case we slide the kth end of the jth handle the resulting diffeomorphisms contain an additional rotation parallel to the nonseparating sphere $S_{(j,k)}$ ($k = 1$ or 2) which is not isotopic to the identity within $D(I \times S_j', \mathrm{rel}\, S'_{(j,1)} \cup S'_{(j,2)})$). This in fact exhausts all ambiguities, since higher rotation numbers in the θ coordinate just result in more rotaations parallel to the spheres, which, within internal diffeomorphisms, are isotopic to the identity for even rotation numbers. Thus we have exactly two isotopically inequivalent definitions for sliding spinorial primes or ends of handles. They differ by rotations of the primes or belt-twists of the handles. Let us now state the fundamental theorems, the proofs of which may be found in the cited literature.

Theorem 3. Every diffeomorphism of Σ is isotopic to a finite sequence of diffeomorphisms built from the four types described above.

Theorem 4 (Rourke and de Sá). Given the prime factorization (1.2), then there is a homotopy equivalence [denoted by $\sim$; $\Omega(\,\cdot\,)$ denotes the space of based loops in (·)]

$$D_F(\Sigma) \sim \left(\prod_{i=1}^{n} D_F(P_i)\right) \times \left(\prod_{j=1}^{l} \Omega SO(3)\right) \times \Omega C \qquad (1.11)$$

The significance of this result lies in the following: It is nontrivial that D_F is a *product* fibration whose fibers are the internal diffeomorphisms. We express this by saying that internal diffeomorphisms do not "interact" with external diffeomorphisms represented by ΩC. Generally, one might have expected a weaker result to hold, namely that D_F is a nonproduct fibration. In fact, had one considered all diffeomorphisms and given up the restriction to those fixing a frame, an analogous result would fail to hold. A counterexample is given by a 3-manifold Σ which is the connected sum of two spinorial primes P_1 and P_2. In $D(\Sigma)$ the rotation parallel to the connected sphere S_1 (considered as an internal diffeomorphism) is isotopic to the rotation parallel to the connecting sphere S_2. That is, there is a path in $D(\Sigma)$ connecting two elements in $D_F(\Sigma)$ which (by spinoriality) cannot be connected by a path running entirely within the internal diffeomorphisms. The fibration thus cannot be a product.

An immediate corollary of the product structure (1.11) is

$$\pi_k(D_F(\Sigma)) = \left(\prod_{i=1}^{n} \pi_k(D_F(P_i))\right) \times \left(\prod_{j=1}^{l} \pi_{k+1}(SO(3))\right) \times \pi_{k+1}(C) \quad \text{for } k \geq 1 \qquad (1.12a)$$

$$\pi_0(D_F(\Sigma)) = \left(\prod_{i=1}^{n} \pi_0(D_F(P_i)) \times \prod_{j=1}^{l} Z_2\right) \tilde{\times}\ \pi_1(C) \qquad (1.12b)$$

where we had to separate the case $k = 0$ from all others for the reason that the homotopy equivalence (1.11) does not imply a direct product structure for the zeroth homotopy groups, as it does for all the higher ones (expressed by (1.12a)). The direct product structure is valid for $k = 0$ only on the level of sets, but not on the group level, when the group structure of each of the sets on the right hand side is taken into account. However, the following facts can be established: (i) the factor in brackets (representing the internal diffeomorphisms) in (1.12b) forms a generally non-normal subgroup, (ii) each spinorial prime P_i has a central Z_2 in $\pi_0(D_F(P_i))$ generated by a rotation of P_i, which together with Z_2's generated by belt twists form a normal $\prod_{i=1}^{m} Z_2$ in $\pi_0(D_F(\Sigma))$, where m is the number of handles plus the number of spinorial primes, (iii) the subgroup of $\pi_0(D_F(\Sigma))$ generated by all slides and rotations of spinorial primes and belt twists (as described under (ii)), is normal. This is how, at this point, (1.12b) should be read. In particular, the symbol $\tilde{\times}$ is a set theoretic Cartesian product, but no direct product on groups.

In (1.12a-b), the space C is still undetermined. It is called the *configuration space of the* 3-*manifold* Σ, and in some sense labels and topologizes the different relative positions of the primes when combined to form Σ. From Theorem 3 it follows that its fundamental group is generated by exchange, spin, and slide diffeomorphisms. Furthermore, it has been shown (Hendriks and McCullough, 1987) that $\Omega C \sim F_l^n \times \Omega C_1$, where F_l is the free group of l generators and F_l^n its n-fold product. It accounts for the slides of the n irreducible primes P_i through the l handles. It would certainly be desirable to continue the factorization to ΩC_1 if possible, but generally not much seems to be known about the detailed structure of C_1. For more general information on C_1 we refer to the literature (Hendriks and Laudenbach, 1984; Hendriks and McCullough, 1987). In the appendix we investigate in more detail the group $\pi_0(D_F(\Sigma))$ for the case where Σ is given by the connected sum of l handles. There we shall be interested in learning how slides interfere with exchange and spin or diffeomorphism by studying small quotient groups with obvious representations. Heuristic arguments in canonical quantum gravity suggest that the different representations of $\pi_0(D_F(\Sigma))$ characterize different sectors in quantum gravity (Isham, 1981). Usually attention is restricted to one-dimensional representations, but this seems unnecessarily restrictive (Giulini, 1992*b*). However, special proposals for the construction of quantum states might be employed to preclude certain sectors (Hartle and Witt, 1988; Giulini and Louko, 1992). In Theorem A of the appendix it is, e.g., shown that restriction to Abelian sectors implies that for an $l > 2$ handle manifold, spins followed by exchanges are necessarily represented trivially. This is a purely kinematical result and independent of any requirement as to how to construct the quantum states. In contrast, the considerations in Hartle and Witt (1988) and Giulini and Louko (1992) made essential use of the no-boundary proposal for the construction of quantum states.

We can now introduce the properties of 3-manifolds which we wish to give information about in this article. We indicate how these properties behave under taking connected sums so that we can eventually restrict attention to prime 3-manifolds. Table I summarizing their properties is presented in Section 2.

• *Chirality*: A manifold is called chiral iff it does not allow for an orientation-reversing self-diffeomorphism. It follows immediately from Theorem 2 that a 3-manifold is nonchiral iff no prime in its prime decomposition is chiral. Chirality for the relevant spherical primes is nicely demonstrated in Witt (1986). Chirality of the flat 3-manifolds R^3/G_i, $i = 3, 4, 5, 6$, may be shown by inspection (D. McCullough, private communication) using results from Lee *et al.* (1993). In Table I, + stands for chiral and − for nonchiral. Chirality is abbreviated by C and is listed in

Table I. Properties of Prime 3-Manifolds.

Prime Π	HC	S	C	N	$H_1(\Pi)$	$\pi_0(D_F(\Pi))$	$\pi_1(D_F(\Pi))$	$\pi_k(D_F(\Pi))$
S^3/D_8^*	+	+	+	−	$Z_2 \times Z_2$	O^*	0	$\pi_k(S^3)$
S^3/D_{8n}^*	+	+	+	−	$Z_2 \times Z_2$	D_{16n}^*	0	$\pi_k(S^3)$
$S^3/D_{4(2n+1)}^*$	+	+	+	+	Z_4	$D_{8(2n+1)}^*$	0	$\pi_k(S^3)$
S^3/T^*	?	+	+	−	Z_3	O^*	0	$\pi_k(S^3)$
S^3/O^*	w	+	+	+	Z_2	O^*	0	$\pi_k(S^3)$
S^3/I^*	?	+	+	−	0	I^*	0	$\pi_k(S^3)$
$S^3/D_8^* \times Z_p$	+	+	+	−	$Z_2 \times Z_{2p}$	$Z_2 \times O^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/D_{8n}^* \times Z_p$	+	+	+	−	$Z_2 \times Z_{2p}$	$Z_2 \times D_{16n}^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/D_{4(2n+1)}^* \times Z_p$	+	+	+	+	Z_{4p}	$Z_2 \times D_{8(2n+1)}^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/T^* \times Z_p$	?	+	+	−	Z_{3p}	$Z_2 \times O^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/O^* \times Z_p$	w	+	+	+	Z_{2p}	$Z_2 \times O^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/I^* \times Z_p$	?	+	+	−	Z_p	$Z_2 \times I^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/D'_{2k(2n+1)} \times Z_p$	+	+	+	+	$Z_p \times Z_{2k}$	$Z_2 \times D_{8(2n+1)}^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/T'_{8\cdot 3m} \times Z_p$	?	+	+	−	$Z_p \times Z_{3m}$	O^*	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$L(p, q_1)$	w	−	+	$(-)^p$	Z_p	Z_2	Z	$\pi_k(S^3)$
$L(p, q_2)$	$w+$	−	+	$(-)^p$	Z_p	$Z_2 \times Z_2$	$Z \times Z$	$\pi_k(S^3) \times \pi_k(S^3)$
$L(p, q_3)$	w	−	−	$(-)^p$	Z_p	Z_2	$Z \times Z$	$\pi_k(S^3) \times \pi_k(S^3)$
$L(p, q_4)$	w	−	+	$(-)^p$	Z_p	Z_2	$Z \times Z$	$\pi_k(S^3) \times \pi_k(S^3)$
RP^3	+	−	−	+	Z_2	1	0	0
S^3	+	−	−	−	0	1	0	0
$S^2 \times S^1$	/	−	−	+	Z	$Z_2 \times Z_2$	Z	$\pi_k(S^3) \times \pi_k(S^2)$
R^3/G_1	/	+	−	+	$Z \times Z \times Z$	$St(3, Z)$	0	$\pi_k(S^3)$
R^3/G_2	/	+	−	+	$Z \times Z_2 \times Z_2$	$Aut_+^{Z_2}(G_2)$	0	$\pi_k(S^3)$
R^3/G_3	/	+	+	+	$Z \times Z_3$	$Aut_+^{Z_2}(G_3)$	0	$\pi_k(S^3)$
R^3/G_4	/	+	+	−	$Z \times Z_2$	$Aut_+^{Z_2}(G_4)$	0	$\pi_k(S^3)$
R^3/G_5	/	+	+	+	Z	$Aut_+^{Z_2}(G_5)$	0	$\pi_k(S^3)$
R^3/G_6	/	+	+	−	$Z_4 \times Z_4$	$Aut_+^{Z_2}(G_6)$	0	$\pi_k(S^3)$
$S^1 \times R_g$	/	+	−	−	$Z \times Z_{2g}$	$Aut_+^{Z_2}(Z \times H_g)$	0	$\pi_k(S^3)$
$K(\pi, 1)_{sl}$	/	+	*	*	$A\pi$	$Aut_+^{Z_2}(\pi)$	0	$\pi_k(S^3)$

the fourth column. An asterisk stands for: has to be decided on a case-by-case analysis.

• *Spinoriality*: A 3-manifold is called spinorial iff a rotation parallel to the boundary sphere of an embedded 3-disc is not in the connected component of D_F, where D_F stabilizes a fixed interior point (and all frames at this point) of the disc. It can be shown (e.g., Giulini, 1992*a*) that a 3-manifold is nonspinorial iff no prime in its prime decomposition is spinorial. In Table I, + stands for spinorial and − for nonspinorial. Spinoriality is abbreviated by S and is listed in the third column.

- *Nuclearity*: A 3-manifold is nuclear iff it is the spacelike boundary of a Lorentz 4-manifold with $SL(2, C)$ spin structure. The necessary and sufficient condition for Σ to be nuclear is that the $\{0, 1\}$-valued, so-called Kevaire semicharacteristic $u(\Sigma)$ is 0 (Gibbons and Hawking, 1992). It can be shown that a 3-manifold is nuclear iff the number of nuclear primes in its prime decomposition is odd. For disconnected Σ one has the following simple rule (Giulini, 1993): Σ is nuclear iff the number of components with even number of nuclear primes in their decomposition is even. In Table I, + stands for nuclear ($u = 0$) and − for nonnuclear ($u = 1$). Nuclearity is abbreviated by N and is listed in the fifth column. There, $(-)^p$ stands for + if p even and − if p odd; an asterisk stands for: has to be decided on a case-by-case analysis.

- *Homology groups of* Σ: The homology groups are merely listed for reasons of completeness. Let A denote the operation of Abelianizing a group and F that of taking the free part of a finitely generated Abelian group. The homology H_* (abbreviating the zeroth to third homology groups as a row-quartuple) is then given in terms of the fundamental group: $H_* = (Z, A\pi_1, FA\pi_1, Z)$. The fundamental group of a connected sum of two 3-manifolds Σ_1 and Σ_2 is the free product (denoted by $*$) of the individual ones: $\pi_1(\Sigma_1 \uplus \Sigma_2) = \pi_1(\Sigma_1) * \pi_1(\Sigma_2)$. It is infinite if neither of the two manifolds is simply connected. For the homology this implies

$$H_*(\Sigma_1 \uplus \Sigma_2) = (Z, H_1(\Sigma_1) \times H_1(\Sigma_2), H_2(\Sigma_1) \times H_2(\Sigma_2), Z)$$

Since $H_2 = FH_1$, it is enough to list H_1 which is done in the sixth column. There, the symbol $A\pi$ denotes the Abelianization of the group represented by π.

- *Homotopy groups of* $D_F(\Sigma)$: In general relativity, the classical configuration space Q satisfies $\pi_k(D_F) = \pi_{k+1}(Q)$, so that we obtain information on its topology by studying the topology of D_F. It is explained in Giulini (1992*a*) in what sense this is equally valid for the configuration space of closed and open universes. Theorem 4 shows how far these groups are fixed by the corresponding ones for the primes. It tells us that the latter ones are contained as subgroups. The zeroth and first homotopy groups of $D_F(\Sigma)$ [resp. the first and second homotopy group of $Q(\Sigma)$] are listed in the seventh and eighth columns, respectively, and the higher ones $\pi_k(D_F(\Sigma))$ for $k \geq 2$ [resp. $\pi_{k+1}(Q(\Sigma))$] are reduced to those of spheres of dimension two and three in the ninth column. Calculations for $\pi_0(D_F(\Pi))$, where Π is a spherical prime, were first presented in Witt (1986). Details of the other calculations are given in Giulini (1992*a*). In the seventh column the symbols $\mathrm{Aut}_+^{Z_2}(G)$ are interpreted as follows: By $\mathrm{Aut}(G)$ [where G is the fundamental group of the prime P, which we identify with $\pi_1(\infty, P)$] we denote the automorphism group of G. It is generated by D_∞ via its action on the fundamental group. $\mathrm{Aut}_+(G)$ denotes the subgroup generated by the orientation-preserving diffeomorphisms $D_\infty^+(P)$. Finally, $\mathrm{Aut}_+^{Z_2}(G)$ denotes a central Z_2 extension thereof (due to spinoriality of P), where the extending Z_2 is generated by a rotation parallel to a sphere whose bounding disc contains ∞. $\mathrm{St}(3, Z)$ is a so-called Steinberg group [see, e.g., paragraph 10 in Milnor (1971) for more information about $\mathrm{St}(n, Z)$], which is a central Z_2 extension of $SL(3, Z)$. It is a perfect group (i.e., its own commutator subgroup). H_g denotes the fundamental group of a genus g Riemann surface.

• *Validity of Hatcher conjecture*: For a particular class of spherical primes (explained below), the information given about $\pi_k(D_F)$ depends on the validity of the so-called Hatcher conjecture. This motivates our indicating its status for the relevant primes within the list. It states that the diffeomorphism group for a spherical prime is (as a topological space) homotopy equivalent to the space of isometries (of the obvious metric). A weak implication thereof (called the weak conjecture) is that these spaces have isomorphic zeroth homotopy group. The calculation of $\pi_0(D_F)$ only depends on the validity of the weak conjecture, whereas those for $\pi_k(D_F)$, $k \geq 1$, depend on the full conjecture. A + indicates validity of the full conjecture; a w the validity of the weak form; and a question mark, that we do not have any information. The Hatcher conjecture is abbreviated by *HC* and is listed for spherical primes in the second column.

2. THE PRIME 3-MANIFOLDS

Table I presents the relevant data for all known prime 3-manifolds except the nonsufficiently large $K(\pi, 1)$. The first column contains their conventional names as already used in the physics literature (e.g., Witt, 1986). The top line names the columns as outlined above. Below this line, the table is divided into three disconnected parts, which we call subtables, the first and third of which are again subdivided into so-called blocks. The first subtable contains the known prime 3-manifolds with the finite fundamental group (which is clearly identical to the set of all known 3-manifolds with finite fundamental group). They are all of the form S^3/G, where G is a finite subgroup of $SO(4)$ with free action on S^3. They are also called spherical primes. The first two blocks contain those S^3/G with G noncyclic, the third $G = Z_p$ for $p > 2$, and the fourth $G = Z_p$ for $p \leq 2$, i.e., the real projective 3-space, $RP^3 = S^3/Z_2$, and the 3-sphere itself.

The first block has $G \subset SU(2)$ and the resulting 3-manifolds are homogeneous. The indexing integer has range $n \geq 1$. In the second block G

is not contained in any $SU(2)$ and the manifolds are not homogeneous. The order of Z_p, i.e., p, is coprime to the order of the group the Z_p is multiplied with and $p > 1$ in the first six cases. In the remaining two cases $p = 1$ is also an allowed value. The other indexing integers have ranges $n \geq 1$, $m \geq 2$, and $k \geq 3$. The third block contains the so-called lens spaces. Since Z_p can act in different, nonequivalent (i.e., not conjugate by a diffeomorphism) ways, they are labeled by $L(p, q)$ with an additional integer q coprime to p. Here, q_1 stands for $q = \pm 1 \bmod p$, q_2 for $q \neq \pm 1 \bmod p$ and $q^2 = 1 \bmod p$, q_3 for $q^2 = -1 \bmod p$, and q_4 for the remaining cases. Among all $L(p, q_2)$ are those of the form $L(4n, 2n - 1)$, $n \geq 2$. For those the $+$ is valid in the HC column and w for the remaining cases. The $L(p, q_1)$ are the only homogeneous lens spaces. Finally, we note that it is a still open conjecture (involving some subconjectures) that this list comprises all 3-manifolds of finite fundamental group (see, e.g., Thomas, 1986, 1988). Presentations of the finite fundamental groups occurring here may be found in Orlik (1972) or Witt (1986).

The second subtable consists of a single member, namely the only nonirreducible prime: $S^2 \times S^3$. The third subtable comprises the irreducible primes with infinite fundamental group which in addition are sufficiently large (SL). They fall into the class of $K(\pi, 1)$ spaces (Eilenberg–MacLane spaces), that is, spaces whose only nonvanishing homotopy group is the first. The condition of being sufficiently large means that these 3-manifolds allow an embedding of a closed Riemannian surface such that the induced homomorphism on the fundamental groups is injective. Physically speaking, a noncontractible loop on the embedded surface is also not contractible within the ambient 3-manifold. In particular, the fundamental group of a SL 3-manifold contains as subgroup the fundamental group of a Riemannian surface. A sufficient condition for an irreducible manifold to be SL is that the first homology group (which is the Abelianization of the fundamental group) is infinite. The reason why we restrict to the subclass of sufficiently large ones is simply that not much seems to be known for general $K(\pi, 1)$'s. Now, the first block contains all 3-manifolds of the form R^3/G, where G is the discrete subgroup of the affine group in 3 dimensions that acts freely and properly discontinuously on R^3. They comprise all flat 3-manifolds (i.e., admitting a flat metric). G_1 is equal to $Z \times Z \times Z$ (i.e., R^3/G_1 is just the 3-torus), and $G_2, \ldots, G_6$ are certain extensions of the groups Z_2, Z_3, Z_4, Z_6, and $Z_2 \times Z_2$, respectively, by G_1 (i.e., $R^3/G_2, \ldots, R^3/G_6$ are further quotients of the 3-torus). The second block contains manifolds of the form $S^1 \times R_g$, where R_g denotes a Riemannian surface of genus g with fundamental group H_g. The third block represents all other sufficiently large $K(\pi, 1)$ primes. For relativists, an interesting property of $K(\pi, 1)$ primes is that no connected sum containing at least one of them admits a Riemannian metric of everywhere positive scalar curvature. Moreover, if it admits a nowhere negative scalar curvature metric, it must in fact be flat, i.e., the 3-manifold must be one of the six ones listed in the first block. This has been proven in Gromov and Lawson (1983) and its significance for general relativity pointed out (Witt, 1987).

APPENDIX

In this appendix we investigate in more detail the diffeomorphisms of the l-fold connected sum of handles:

$$\Sigma = \biguplus_{i=1}^{l} S^1 \times S^2 \tag{A1}$$

Its fundamental group is given by the l-fold free product of Z:

$$\pi_1(\Sigma) = \mathop{\ast}_{i=1}^{l} Z =: F_l \tag{A2}$$

where F_l just stands for the free group on l generators. We visualize the generator g_i of the ith Z as a smooth, nonintersecting, and oriented curve which starts from some basepoint, enters the ith handle through $S_{(i,1)}$, leaves it through $S_{(i,2)}$, and returns to the basepoint. The direction defined by g_i is called positive. By $g_i g_j$ we denote a curve that first traverses the ith and then the jth handle in positive directions.

According to equation (1.12b) and the following remarks, one has (McCullough, 1986)

$$\pi_0(D_F(\Sigma)) = \left(\bigoplus_{i=1}^{l} Z_2 \right) \tilde{\times}\ \pi_1(C_1) \tag{A3}$$

where the ith Z_2 is generated by a belt-twist of the ith handle.

As remarked earlier, $\pi_1(C_1)$ is generated by slides, exchanges, and spins. In paragraph 4.3 of Laudenbach (1974) it is shown that the following sequence is exact:

$$0 \to \bigoplus_{i=1}^{l} Z_2 \to \pi_0(D^+_\infty(\Sigma)) \to \mathrm{Aut}(F_l) \to 1 \tag{A4}$$

where $\bigoplus_{i=1}^{l} Z_2$ is the same as above (i.e., the ith Z_2 is generated by a belt-twist of the ith handle). Now, the handle manifold $S^1 \times S^2$ is nonspinorial, so that Σ is also nonspinorial. But for nonspinorial 3-manifolds one has (Giulini, 1992*a*; Witt, 1986)

$$\pi_0(D_F) \cong \pi_0(D^+_\infty) \tag{A5}$$

Together (A3)–(A5) imply

$$\pi_0(D_F(\Sigma)) = \left(\bigoplus_{i=1}^{l} Z_2 \right) \times \mathrm{Aut}(F_l) \tag{A6}$$

that is,

$$\pi_1(C_1) \cong \mathrm{Aut}(F_l) \tag{A7}$$

In quantum gravity one is, e.g., interested in some of the representation properties of $\pi_0(D_F)$, which we can now investigate. We are interested in the question of how slides interact with the other operations, in particular the exchanges. The reason for this is that slides generate those diffeomorphisms which mix the interior and exterior of primes (as explained in Section 1) and are thus harder to interpret physically, at least in an approximation where the primes are treated as individual particlelike entities (geons). In view of this, one might, e.g., be interested in the following question: Under what conditions may we forget about slides? A first simple answer for the example considered here is provided by Theorem A below.

Since internal diffeomorphisms (here the belt-twists) can be given representation independent of the rest, we may restrict attention to the group $\Gamma_l := \mathrm{Aut}(F_l)$. We will follow Chapter 7 of Coxeter and Moser (1965) in our description of the presentations for Γ_l. Γ_l can be generated by four generators: P, O, Q, and U, whose action on the g_i is given by

$$P: \quad [g_1, g_2, g_3, \ldots, g_l] \mapsto [g_2, g_1, g_3, \ldots, g_l] \tag{A8a}$$

$$Q: \quad [g_1, g_2, g_3, \ldots, g_l] \mapsto [g_2, g_3, g_4, \ldots, g_1] \tag{A8b}$$

$$O: \quad [g_1, g_2, g_3, \ldots, g_l] \mapsto [g_1^{-1}, g_2, g_3, \ldots, g_l] \tag{A8c}$$

$$U: \quad [g_1, g_2, g_3, \ldots, g_l] \mapsto [g_1 g_2, g_2, g_3, \ldots, g_l] \tag{A8d}$$

Their "physical" interpretation is as follows: P exchanges handle 1 and handle 2, Q exchanges all l handles in cyclic order, O spins handle 1, and U slides the second end of the first handle through the second handle in a negative direction. P and Q alone generate the permutation group of l objects which is a sub- but no factor group of Γ_l, and P, Q, O generate an order $2^l l!$ subgroup with obvious interpretation (it may be characterized as the group of rearrangements of l books in a shelf with no orientations given to the backs). The element $(OU)^2$ $[=(UO)^2]$ represents a slide of the whole of the first handle through handle two. If, in addition to the relations given below, one imposes the relation $(OU)^2 = E$, one obtains a presentation of the group $GL(l, Z)$ which is a factor group of Γ_l (Coxeter and Moser, 1965).

We shall now give the relations for the generators P, Q, O, and U with E the identity. Given them, we then study some quotient groups by imposing additional relations $R_i(P, Q, O, U) = E$ by hand, which gives us a presentation of Γ_l/N_R, where N_R is the smallest normal subgroup in Γ_l containing the elements $R_i(P, Q, O, U)$. The reason for this is that a representation $\rho: \Gamma_l \to GL(l, C)$ satisfying $R_i(\rho(P), \rho(Q), \rho(O), \rho(U)) = 1$ comes from a representation of Γ_l/N_R, which, in the cases we choose, is very simple. We can thus immediately give all the representations of Γ_l which satisfy these relations.

Γ_1 is uninteresting and just given by Z_2, generated by the single spin O. For Γ_2 one has $P = Q$ and the relations for the remaining generators are

$$P^2 = O^2 = (PO)^4 = (POPU)^2 = (POU)^3 = E, \qquad (OU)^2 = (UO)^2 \tag{A9}$$

Let us look at this case first before going to the general case.

● *Abelian representations*: The presentation of the Abelianization $A\Gamma_2$ of Γ_2 is easily obtained (all generators commute) and given by

$$P^2 = O^2 = U^2 = POU = E \tag{A10}$$

which is just the group $Z_2 \times Z_2$ generated by P (left Z_2) and O (right Z_2) and where U generates the diagonal Z_2. For an Abelian representation this implies that any of the generators P, O, U is nontrivially represented precisely if one of the others is. There is no P–Q correlation unless one imposes $U = E$. This is also true generally since the factor group $U = E$ is Abelian, as we show below. One easily checks from (A9) that taking any of the generators P, O, or U to commute with the other two already implies commutativity of all generators. Moreover, it is in fact sufficient to require exchanges to commute with slides only. Proof: If P and U commute,

$$(POPU)^2 = P(OU)^2 P = E \quad \Rightarrow \quad (OU)^2 = E$$

and

$$(POU)^3 = POP(UO)^2 PU = POU = E \quad \Rightarrow \quad PU = O$$

so that P also commutes with O.

- *Slides represented trivially*: Setting $U = E$ yields

$$P^2 = O^2 = (PO) = E \tag{A11}$$

which is just Z_2 generated by $P = O$. Thus representing U trivially leads to an Abelian representation with $P-O$ correlation.

- *P–O correlating representations*: Setting $P = O$ yields

$$P^2 = U^3 = (PU)^2 = E \tag{A12}$$

which is the presentation of the permutation group S_3 of three objects [e.g., $P = (12)$ and $U = (123)$] or, equivalently, of D_6, the dihedral group of order 6 which describes the symmetries of an equilateral triangle (P generates reflections about a symmetry axis, U a rotation by $2\pi/3$). There are two nontrivial representations, the one-dimensional one, R_1, and the two-dimensional one, R_2:

$$\begin{aligned} R_1 &: \quad P \mapsto -1, \quad U \mapsto 1 \\ R_2 &: \quad P \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U \mapsto \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \end{aligned} \tag{A13}$$

R_2 shows that there are $P-O$ correlating representations which also represent U nontrivially.

- *Spins represented trivially*. Setting $O = E$ yields

$$P^2 = U^2 = (PU)^3 = E \tag{A14}$$

which reduces to (A12) by replacing $U \to PU$. Correspondingly, the two nontrivial representations follow from (A14). This shows how slides can interact with exchanges without involving spins.

Next we turn to the general case $l > 2$. A complete list of relations is given below (the relations are not independent). $A \leftrightarrow B$ means that A and B commute.

$$P^2 = E \tag{A15a}$$

$$(QP)^{n-1} = Q^n \tag{A15b}$$

$$P \leftrightarrow Q^{-i}PQ^i \quad \text{for} \quad 2 \le i \le l/2 \tag{A15c}$$

$$O^2 = E \tag{A15d}$$

$$O \leftrightarrow Q^{-1}PQ \tag{A15e}$$

$$O \leftrightarrow QP \tag{A15f}$$

$$(PO)^4 = E \tag{A15g}$$

$$U \leftrightarrow Q^{-2}PQ^2 \quad \text{for} \quad l > 3 \tag{A15h}$$

$$U \leftrightarrow QPQ^{-1}PQ \tag{A15i}$$

$$U \leftrightarrow Q^{-2}OQ^2 \tag{A15j}$$

$$U \leftrightarrow Q^{-2}UQ^2 \quad \text{for} \quad l > 3 \tag{A15k}$$

$$U \leftrightarrow OUO \tag{A15l}$$

$$U \leftrightarrow PQ^{-1}OUOQP \tag{A15m}$$

$$U \leftrightarrow PQ^{-1}PQPUPQ^{-1}PQP \tag{A15n}$$

$$(PQ^{-1}UQ)^2 = UQ^{-1}UQU^{-1} \tag{A15o}$$

$$U^{-1}PUPOUOPO = E \tag{A15p}$$

$$(POPU)^2 = E \tag{A15q}$$

● *Abelian representations*: In this case (A15) boils down to the following relations for commuting generators:

$$P^2 = O^2 = PO = P^{l-1}Q^{-1} = U = E \tag{A16}$$

which is the presentation of Z_2 generated by $P\ (=O)$. Q equals E for l odd and P for l even. The difference of this case from $l = 2$ lies in the condition $U = E$ which followed from (A15o) in view of (A15a). In fact, (A15o) implies $U = E$ already from the commutativity of U with P and Q. Abelian representations (equivalently, representations for which slides and exchanges commute) thus necessarily represent slides trivially.

● *Slides represented trivially*: If we set $U = E$, (A15p) and (A15a) imply $P = O$, (A15f) says that P and Q commute, and (A15b) then implies that $P^{l-1} = Q$, i.e., that $Q = E$ for l odd and $Q = P$ for l even. We thus obtain the group Z_2 generated by $P\ (=O)$.

● *P–O correlating representations*: If we impose $P = O$, (A15f) implies $P \leftrightarrow Q$, (A15i) implies $U \leftrightarrow Q$, and (A15j) implies $U \leftrightarrow P$, so that all generators commute.

We can summarize these points in the following theorem.

Theorem A. Let ρ be a representation for Γ_l, $l \geq 3$. The following conditions on ρ are equivalent:

(a) Slides and exchanges commute.
(b) ρ is Abelian.
(c) Slides are represented trivially.
(d) ρ correlates P and O, i.e., $\rho(P) = \rho(O)$.

ACKNOWLEDGMENTS

I thank D. McCullough for pointing out Lee *et al.* (1993) and K. B. Lee for providing me with a copy of this reference.

REFERENCES

Aneziris, C., Balachandran, A. P., Bourdeau, M., Jo, S., Ramadas, T. R., and Sorkin, R. (1989), *International Journal of Modern Physics A*, **4**, 5459–5510.

Coxeter, H. S. M. and Moser, W. O. J. (1965). *Generators and Relations for Discrete Groups*, 2nd edition, Springer-Verlag, Berlin, Göttingen, Heidelberg, New York.

Friedman, J., and Sorkin, R. (1980). *Physical Review Letters*, **44**, 1100–1103.

Gibbons, G. W., and Hawking, S. W. (1992). *Communications in Mathematical Physics*, **148**, 345–352.

Giulini, D. (1992*a*). On the configuration space topology in general relativity, Preprint, Freiburg THEP-92/32 and gr-qc 9301020, submitted for publication.

Giulini, D. (1992*b*). *Communications in Mathematical Physics* **148**, 353–357.

Giulini, D. (1993). Quantum mechanics on spaces with finite non-Abelian fundamental group, in preparation.

Giulini, D., and Louko, J. (1992). *Physical Review D*, **46**, 4355–4364.

Gromov, M., and Lawson, B. (1983). *Institut des Hautes Études Scientifique Publicationes Mathematiques*, **58**, 294–408.

Hartle, J., and Witt, D. (1988). *Physical Review D*, **37**, 2833–2836.

Hempel, J. (1976). *3-Manifolds*, Princeton University Press, Princeton, New Jersey.

Hendriks, H., and Laudenbach, F. (1984). *Topology*, **23**, 423–443.

Hendriks, H., and McCullough, D. (1987). *Topology and its Application*, **26**, 25–31.

Isham, C. J. (1981). *Physics Letters B*, **106**, 188–192.

Laudenbach, F. (1974). *Asterisque*, **12**, 1–137.
Lee, K. B., Shin, J., and Yokura, S. (1993). Free actions of finite Abelian groups on the 3-torus, University of Oklahoma preprint.
McCullough, D. (1986). *Geometric and Algebraic Topology* (Warsaw), **18**, 61–76.
Milnor, J. (1971). *Introduction to Algebraic K-Theory*. Princeton University Press, Princeton, New Jersey.
Orlik, P. (1972). *Seifert Manifolds*, Springer-Verlag, Berlin.
Rourke, C., and de Sá, C. (1979). *Bulletin of the American Mathematical Society*, (New Series) **1**, 251–254.
Sorkin, R. (1989). Classical topology and quantum phases: Quantum geons, in *Geometrical and Algebraic Aspects of Nonlinear Field Theory*, S. De Filippo, M. Marinaro, G. Marmo, and G. Vilasi, eds., Elsevier, North-Holland.
Thomas, C. B. (1986). *Elliptic Structures on 3-Manifolds*, Cambridge University Press, Cambridge.
Thomas, C. B. (1988). *Bulletin of the London Mathematical Society*, **20**, 65–67.
Witt, D. (1986). *Journal of Mathematical Physics*, **27**, 573–592.
Witt, D. (1987). Topological obstructions to maximal slices, Santa Barbara preprint, UCSB-1987.